Georg Fantner

Week 3
Strain and stress in one dimension

1. Treatment of bars like springs
2. Stress in axially loaded bars
3. Microscopic equilibrium
4. Superposition principle

Stress-Strain relationships: Hooke's law

What is the relationship between the applied load and the deformation of a structure?

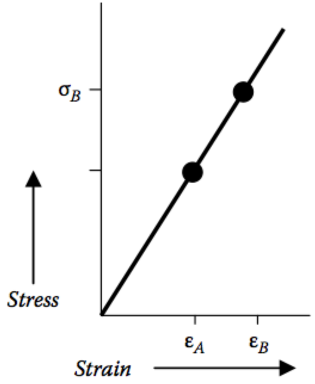
EPFL

Stress Strain relationships

- For each incremental stress there is a proportional increase of strain $\sigma \propto \varepsilon$

- Hooke's law for normal stress: $\sigma = E \cdot \varepsilon$
- *E = "Youngs modulus"* or *"elastic modulus"*

- Hooke's law for shear stress: $\tau = G \cdot \gamma$
- *G="shear modulus"* or *"modulus of rigidity"*



- Proportionality limit: the limit of where the increase in stress becomes non-linear with the increase in strain

Assumptions we are making in the simple form of Hooke's law

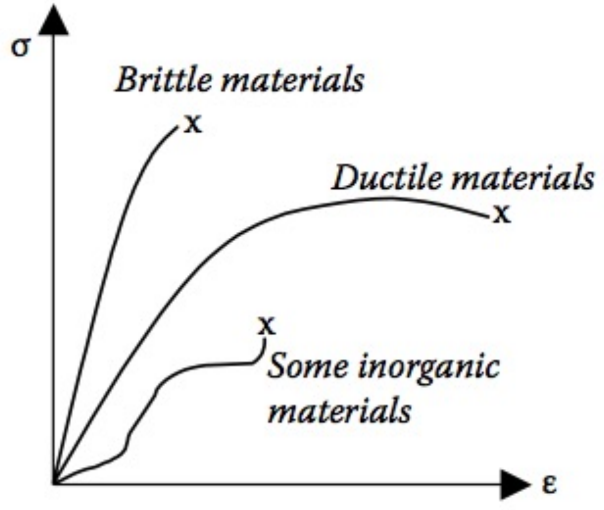
The material is homogeneous: E and G do not vary from point to point

The material isotropic: E and G are invariant with respect to any rotation of the coordinate system

There is no effect of temperature on the mechanical properties E and G

Materials classification:

- Ductile materials: materials that can withstand a large amount of strain without significant increase in stress (rubber, polymers, skin, low carbon steel)
- Brittle materials: materials that experience a huge increase in stress for even small strains. These materials will fail abruptly after small amounts of deformation (ceramics, silicon, cast iron, concrete)
- Some materials have different properties in different directions (wood, bone, carbon composites). Such materials CAN NOT be treated with the simple form of Hooke's law.



Deformation in axially loaded bars

- Hooke's Law:

$$\sigma = E \cdot \varepsilon$$

$$\frac{P}{A} = E \cdot \frac{\delta}{L}$$

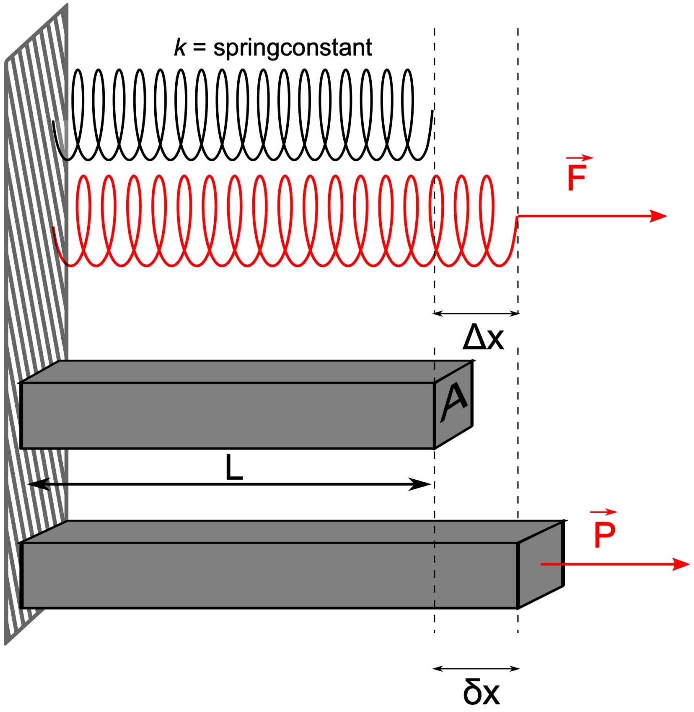
- Strain in axially loaded bar with uniform cross-section:

$$\delta = \frac{PL}{AE} \qquad P = \underbrace{\frac{AE}{L}}_{\text{stiffness of bar}} \cdot \delta$$

- Strain in axially loaded bar with arbitrary material or cross-section:

$$\delta = \int_0^L \frac{P(x)}{A(x)E(x)} dx$$

Treatment of bodies like springs



From the axially loaded bar we have seen the communality of the load extension curve to the basic force extension curve of a spring:

	Spring	Axially loaded bar
Hooke's Law	$F = k \cdot \Delta x$	$P = \frac{AE}{L} \cdot \delta$
Spring constant	k	$k = \frac{AE}{L}$

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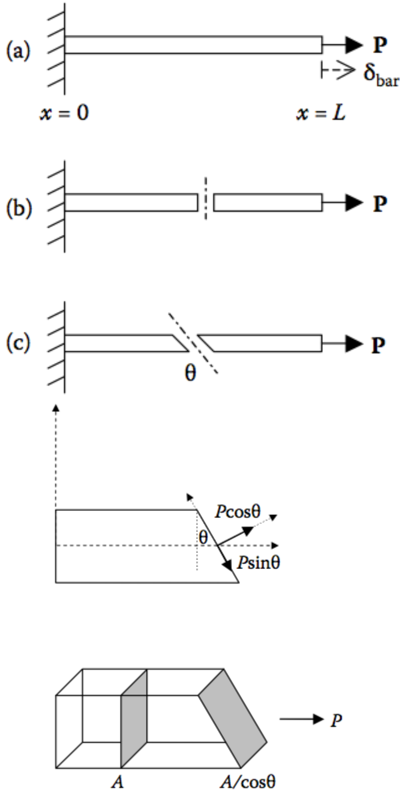
Stress in axially loaded bars

What happens inside the bar when it is stretched?

- Axially loaded bar fixed at x=0 and loaded by force P at x=L (figure a)
- Apply method of section normal to bar axis (figure b)

$$\sigma \equiv \frac{P}{A}$$

- Apply method of section angled to bar axis (figure c)



Angular dependence of stress

$$\sigma_\theta = \frac{force}{area} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta$$

$$\tau_\theta = -\frac{P\sin\theta}{A/\cos\theta} = -\frac{P}{A}\sin\theta\cos\theta$$

Stress in axially loaded bars

Directions of maximum stress

- Maximum normal stress:

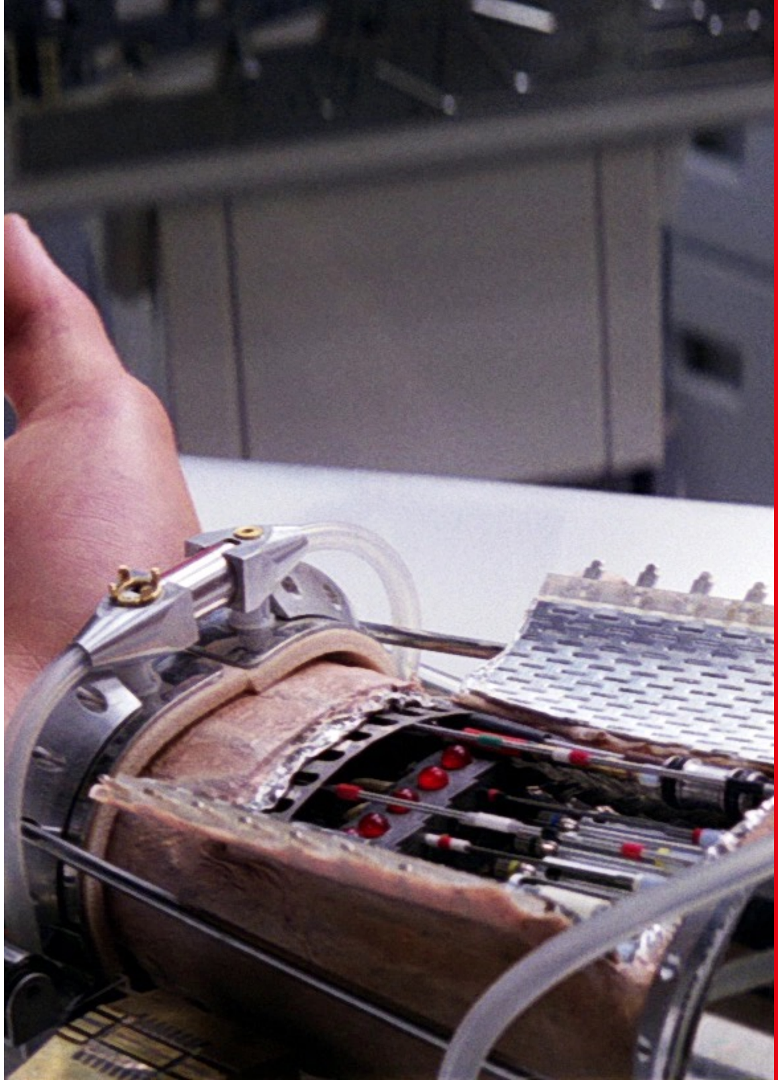
$$\sigma_\theta = \frac{P}{A}\cos^2\theta \rightarrow \boxed{\sigma_{max} = \frac{P}{A}} @\, \theta = 0$$

- Maximum shear stress:

$$\frac{d}{d\theta}\tau_\theta = \frac{d}{d\theta}\left(-\frac{P}{A}\sin\theta\cos\theta\right) = 0 \rightarrow \theta = 45^o$$

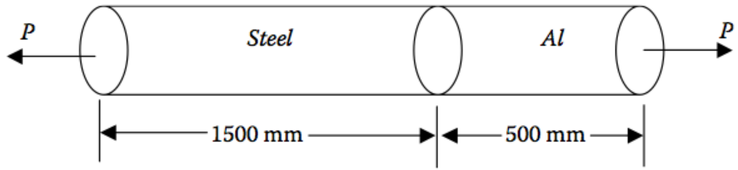
$$|\tau_{max}| = \frac{P}{A}\sin(\pi/4)\cos(\pi/4)$$

$$\boxed{|\tau_{max}| = \frac{P}{2A} = \frac{\sigma_{max}}{2}}$$



Example: elongation of a steel bar

You need to design Luke Skywalkers cybernetic arm. Each round steel pull-bar in the arm has to carry a load of 500N. The steel bar (E=200GPa) has a length of 10cm. The bar is not allowed to stretch more than 8um, nor must it exceed a stress of 150MN/m^2. What is the minimum diameter that each bar must have?



Example: Multi-material composite bar

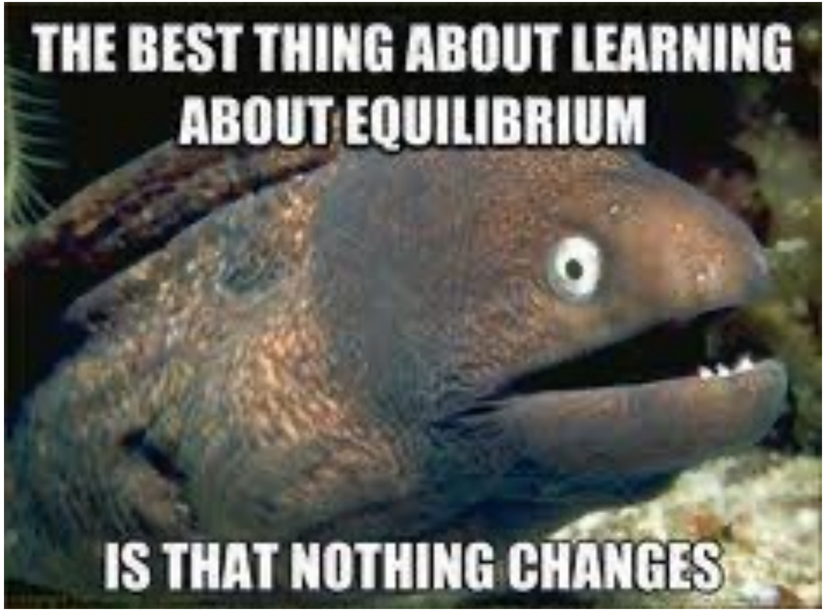
A solid bar 50 mm in diameter and 2000 mm long consists of a steel and an aluminum section, as shown below. When an axial force *P* is applied to the system, a strain gauge attached to the aluminum indicates an axial strain of 0.000873 m/m. The elastic modulus of steel is =E_{ST}=200GPa, E_{ALU}=70GPa.

(a) Determine the magnitude of applied force *P,* and

(b) if the system behaves elastically, find the total elongation of the bar.

Microscopic equilibrium

We have defined strain in microscopic terms, we have defined Hookes law in terms of stress and strain, now we need to define equilibrium in microscopic terms.



What kind of forces can act on an object?

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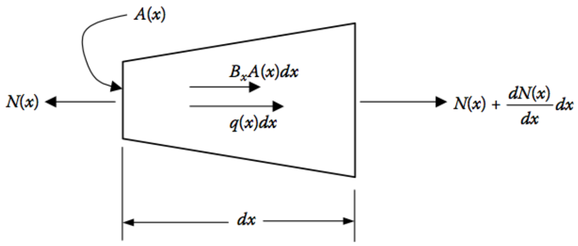
Point	Point loads P: A discrete force acting at one point and one point only
Distributed	Distributed axial loads q(x): force that acts per unit length
Body	Body forces B(x): force that acts per volume (e.g. gravity, magnetic field etc.)

Microscopic equilibrium analysis

Calculation of the stress at an arbitrary point in the structure acting on an infinitesimal element dV=A(x)dx

The *internal* axial force N(x) balances both *external* forces (distributed axial load q(x), and axial body force $B_{x)}$



$$\left\{N(x) + \frac{dN(x)}{dx} \cdot dx\right\} - N(x) + q(x)dx + B_x A(x)dx = 0$$

$$\boxed{\frac{dN(x)}{dx} + q(x) + B_x A(x) = 0}$$

First order, ordinary differential equation for the axial normal stress

The three equations

In structural mechanics, we (always) rely on these 3 equations:

- Equilibrium equation:
  ensures that all forces are in equilibrium

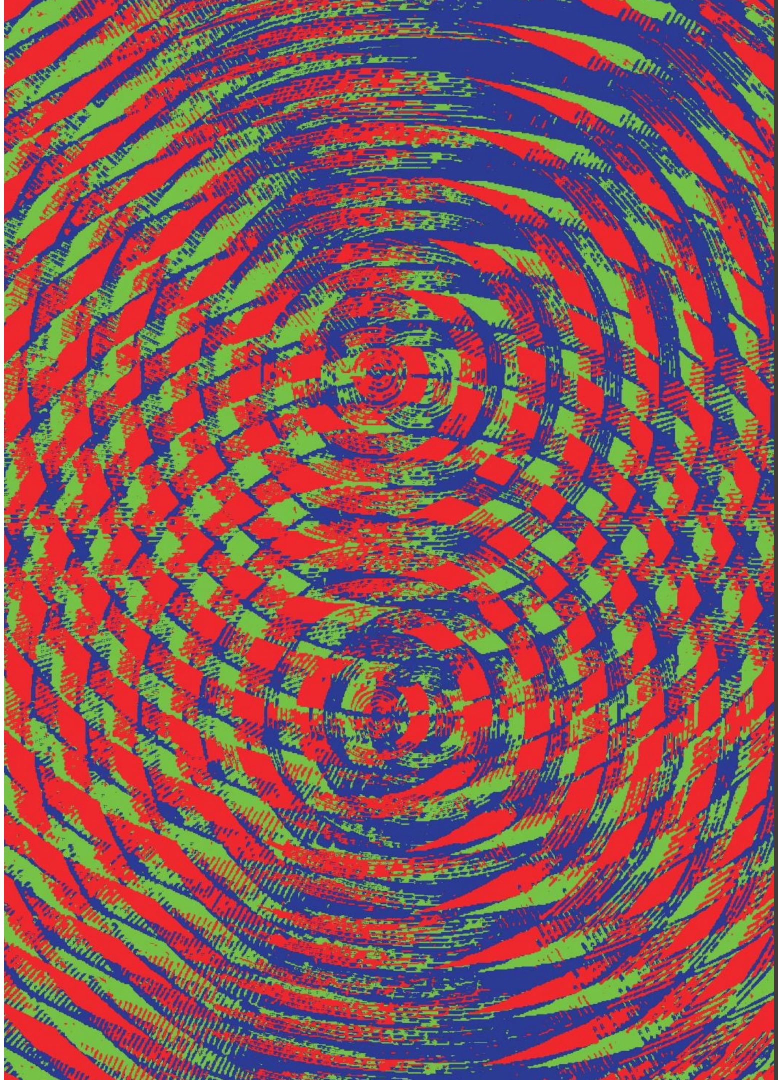
$$\frac{dN(x)}{dx} + \sum_i P_i \delta(x - x_i) + B_x A(x) = 0$$

- Constitutive equation:
  Relates two quantities with materials specific properties:

$$E = \frac{\sigma}{\varepsilon}$$

- Kinematic Equation:
  relates strain (ε) to displacement (u):

$$\varepsilon(x) = \frac{du(x)}{dx}$$



Superposition principle

The effect of the sum of forces is equal to the sum of the effects of the individual forces.

$$F(x_a + x_b) = F(x_a) + F(x_b)$$

$$F(a * x_a) = a * F(x_a)$$

This principle is only valid whenever there exists a linear relationship between the external loads and the structural displacement! In our case we deal only with small displacements where Hooke's law is valid. So the superposition principle is valid.

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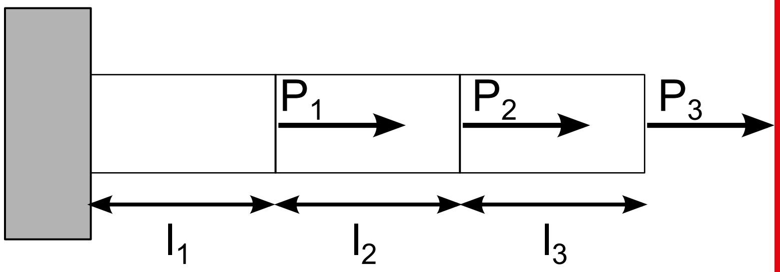
Example: superposition



A uniform bar with cross-section A and elastic modulus E is loaded at 3 points along its axis as shown below. What is the total elongation of the bar?